Ring Theory 11

17 April 2024 16:04

Euclidean Domains :-

It is a integral domain R which can be equipped with a function, $d : R \setminus \{0\} \longrightarrow \mathbb{N}$ such that $\forall a \in R, b \neq 0, b \in R$ we get $a = bq + r$ for some $q, r \in R$ with $r = 0$ or $d(r) < d(b)$

Example :-

① $\mathbb{Z}$ with $d(n) = |n|$ $\qquad \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N}$

$$\{|\mathbb{Z} \setminus \{0\}|\}$$

$\forall a \in \mathbb{Z}, b \neq 0$ and $b \in \mathbb{Z}$

$a = bq + r$ $\Rightarrow r = 0$ or $d(r) < d(b)$

remainder in $\mathbb{Z}$

quotient in $\mathbb{Z}$

② In any field F, $F[X]$ with $d = \deg(f)$ is a Euclidean domain

③ For any ring R, $d = 1$

$R \setminus \{0\} \rightarrow \mathbb{N}$

$\forall a \in R, b \neq 0, b \in R$ we get $a = bq + r$

$d(r) = d(b) = 1$

$\Rightarrow r = 0$

$\Rightarrow a = bq$ for some $q \in R$

Proposition :- In a Euclidean domain every ideal is a principle.

Proof :- R is an Euclidean domain and I be an ideal of R.

Then either $I = \{0\} = (0)$ or we can take $a \neq 0$ and $a \in I$ with minimum $d(a)$. Then for any $b \in I$ we get $b = aq + r$ with $r = 0$ or $d(r) < d(a)$.

But, $r = q - ba \in I \Rightarrow d(r) > d(a)$

$\Rightarrow \Leftarrow$ contradiction.

$\Rightarrow r = 0 \quad \Rightarrow a \mid b$ and $I = (a)$

$\Rightarrow I$ is principle

Q> Every element of the ring $\mathbb{Z}[\sqrt{-2}]$ can be factorized into primes, i.e., irreducibles and the factorization is essentially unique.

Ans:- $d : \mathbb{Z}[\sqrt{-2}] \setminus \{0\} \longrightarrow \mathbb{N}$

$$\underbrace{d(a+b\sqrt{-2})}_{\in \mathbb{Z}\sqrt{-2}} = (a+b\sqrt{-2})(a-b\sqrt{-2})$$

$$= \underbrace{a^2+2b^2}_{\in \mathbb{N}} . \text{ as } a, b \in \mathbb{Z}$$

$\forall\ a+b\sqrt{-2} \in \mathbb{Z}[\sqrt{-2}]$ we get,

$$a+b\sqrt{-2} = q(c+d\sqrt{-2}) + r$$

for $q = 1 \in \mathbb{Z}\sqrt{-2}$ and $r = 0$ we get it is Euclidean Domain

If $q \neq 1$ then,

$$d(r) = d\left(a+b\sqrt{-2} - q(c+d\sqrt{-2})\right)$$

$$= d\left(a - qc + (b-qd)\sqrt{-2}\right)$$

$$= (a-qc)^2 + 2(b-qd)^2$$

d can be extended to $\mathbb{Q}$ as well by some norm

$$\frac{a+b\sqrt{-2}}{c+d\sqrt{-2}} = \left(\frac{a}{c^2+2d^2} + \frac{b\sqrt{-2}}{c^2+2d^2}\right)(c-d\sqrt{-2}) = s+t\sqrt{-2}$$

$s, t \in \mathbb{Q}$

Let $x+z\sqrt{-2}$ such that $x - s \leq 1/2$ and $z - t \leq 1/2$

$$a+b\sqrt{-2} = (x+z\sqrt{-2})(c+d\sqrt{-2}) + r$$

$$\Rightarrow r = a+b\sqrt{-2} - (xc + xd\sqrt{-2} + zc\sqrt{-2} - 2zd)$$

$$\Rightarrow r = a + b\sqrt{-2} - (xc + xd\sqrt{-2} + zc\sqrt{-2} - 2zd)$$

$$d(r) = d\left((c+d\sqrt{-2})(s+t\sqrt{-2}) - (c+d\sqrt{-2})(x+z\sqrt{-2})\right)$$

$$= d\left((c+d\sqrt{-2})(s-x+(t-z)\sqrt{-2})\right)$$

$$= d(c+d\sqrt{-2})\, d(s-x+(t-z)\sqrt{-2})$$

$$= d(c+d\sqrt{-2})\left((s-x)^2 + 2(t-z)^2\right)$$

$$= d(c+d\sqrt{-2})\left(\frac{1}{4} + 2\left(\frac{1}{4}\right)\right)$$

$$\Rightarrow d(r) < d(c+d\sqrt{-2})$$

So $\mathbb{Z}[\sqrt{-2}]$ is an Euclidean Domain.

Def:- Suppose that R is an integral domain. Any function $d: R \to \mathbb{N} \cup \{0\}$ with $d(0) = 0$ is a norm. If $d(a) > 0$, $\forall a \in R \setminus \{0\}$ then d is a positive norm.

Definition:- Let R be a comm. ring and let $a, b \in R$ with $b \neq 0$. Then,

(i) a is a multiple of b if there is $c \in R$ such that $a = bc$.

(ii) A greatest common divisor of a and b (if it exists) is an element $g \in R$ satisfying $g|a$ and $g|b$ and if $\exists h \in R, h \neq g$ and $h|a$ and $h|b \Rightarrow g|h$

Proposition:- Suppose that R is an integral domain

Proposition:- Suppose that R is an integral domain and $g, h \in R$. If $(g) = (h)$ then there is a unit $u \in R^{\times}$ such that $g = hu$. In particular if g and h are greatest common divisor of a and b then $g = hu$ for some $u \in R^{\times}$

Definition:- Suppose that R is an integral domain. Let $\tilde{R} = R^{\times} \cup \{0\}$. Then $u \in R \setminus \tilde{R}$ is a universal side divisor if $\forall x \in R$, $\exists z \in \tilde{R}$ such that $u \mid (x - z)$, i.e., there is $q \in R$ and $z \in R$ such that $x = qu + z$

Proposition:- Let R be an integral domain and not a field. If R is an Euclidean domain then R has universal side divisors

Example:- $\mathbb{Z}[\sqrt{-2}]$ is an integral domain

$$(\mathbb{Z}[\sqrt{2}])^{\times} = \{1, -1, i, -i\}$$

$$(\widetilde{\mathbb{Z}[\sqrt{2}]}) = \{0, 1, -1, i, -i\}$$

$x = a + b\sqrt{-2} \in \mathbb{Z}(\sqrt{-2})$ $\qquad u \in (\mathbb{Z}[\sqrt{-2}] \setminus \tilde{\mathbb{Z}}[\sqrt{-2}])$

$z \in (\widetilde{\mathbb{Z}[\sqrt{2}]})$ $\qquad$ to prove or not prove $\quad u \mid (x - z) \quad \forall x.$